Understanding Spatial Spillovers & Feedbacks

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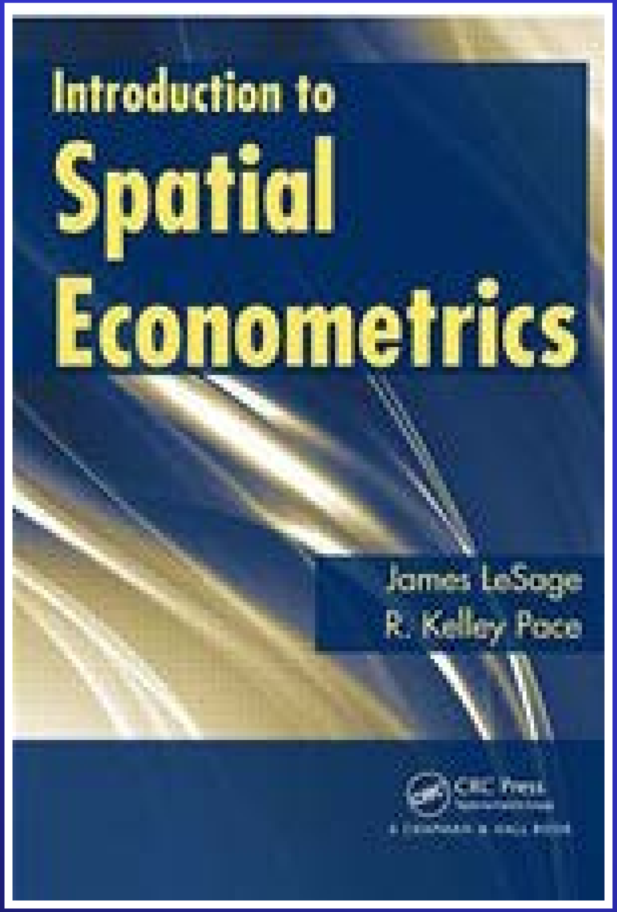
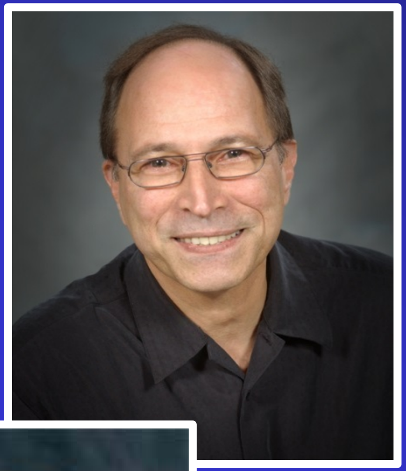
Spatial Effects

Spatial spillovers
Spatial feedbacks





Spatial Lag Model (SAR model)

$$y = X\beta + \rho W y + \varepsilon$$

Model assumes large-scale spatial heterogeneity is handled by $\boldsymbol{X\beta}$; remaining small-scale (localized) spatial dependence is handled as an autoregressive, interactive, effect through $\boldsymbol{Wy}$ & ρ

Rearranging the terms in this model we obtain the reduced form:

$$y = X\beta + \rho Wy + \varepsilon$$

$$y - \rho Wy = X\beta + \varepsilon$$

$$(I - \rho W)y = X\beta + \varepsilon$$

$$y = (I - \rho W)^{-1} X\beta + (I - \rho W)^{-1} \varepsilon$$

Not an easy model to explain in words

Fortunately…

$$(I-\rho W)^{-1} = I + \rho W + \rho^2 W^2 + \rho^3 W^3 \ldots$$

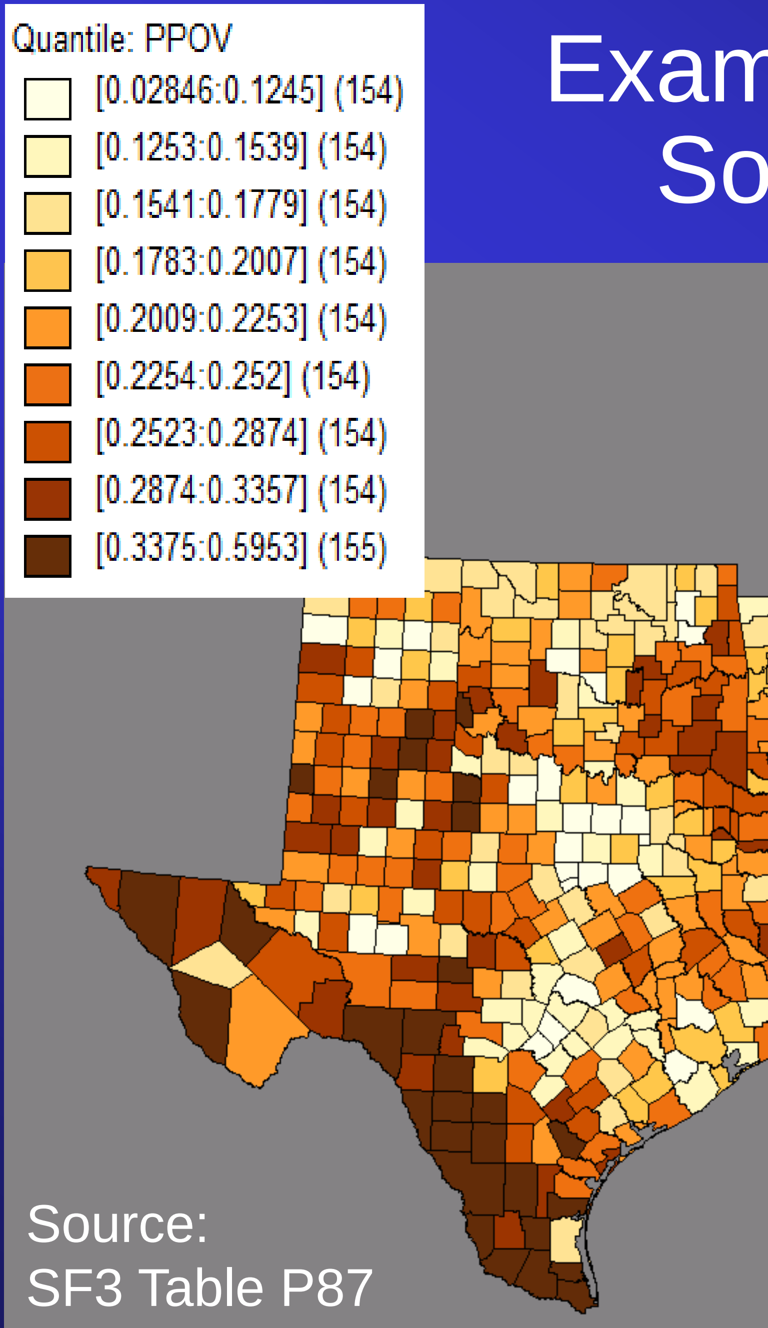
So we can re-express our model…

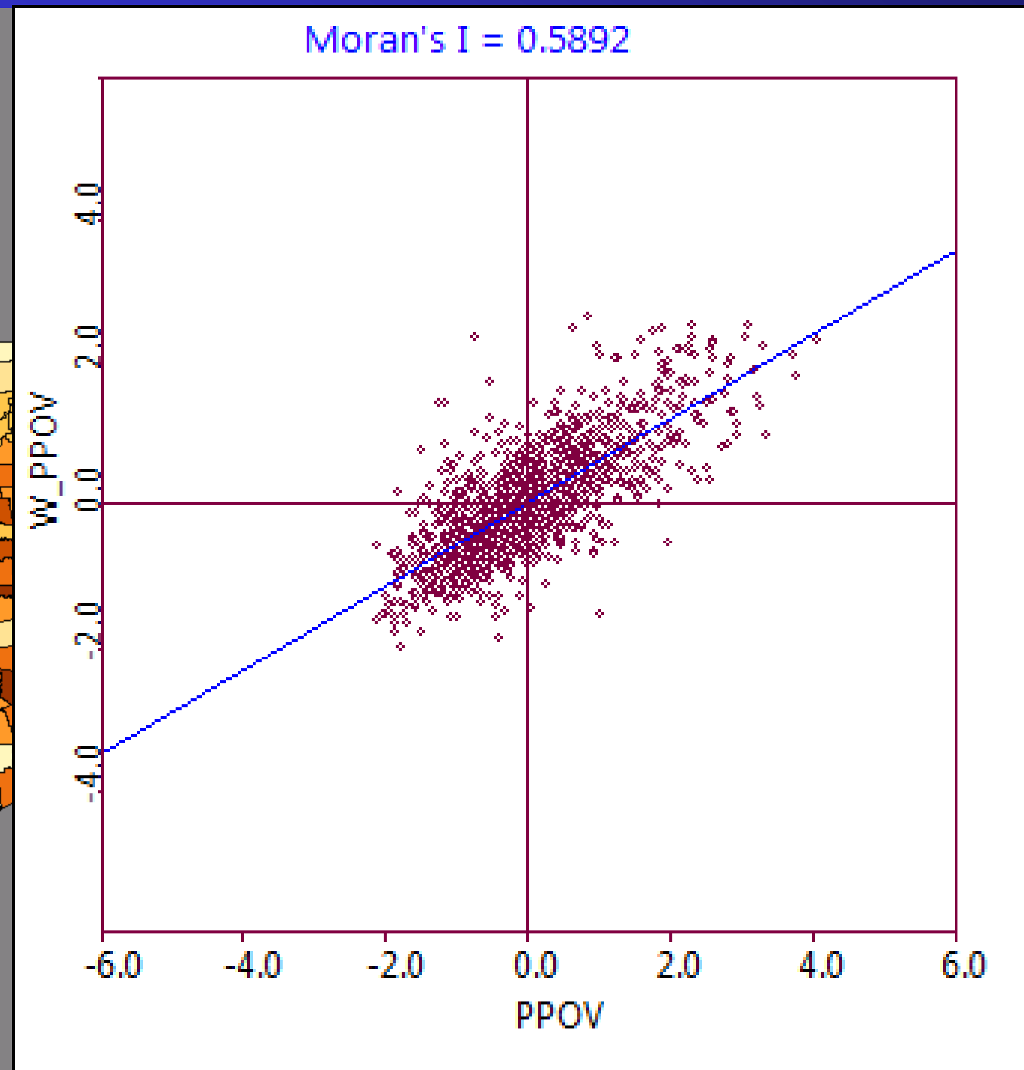
$$y = \left[I + \rho W + \rho^2 W^2 + \ldots\right] X\beta + \left[I + \rho W + \rho^2 W^2 + \rho^3 W^3 + \ldots\right]\varepsilon$$

and taking expected values…

$$\hat{y} = \left[I + \rho W + \rho^2 W^2 + \ldots\right] X\beta$$

Example: Counties in U.S. South, 2000 Census





Source:
SF3 Table P87

Standard Linear Model (transformed variables)

$$
\begin{aligned}
&\mathit{Child\ Poverty\ Rate} = -0.062 \\
&+ (0.387)(\mathit{Proportion\ Female\text{-}Headed\ HH}) \\
&+ (0.828)(\mathit{Unemployment\ Rate}) \\
&+ (-0.142)(\mathit{Higher\ Education\ Rate}) \\
&+ \varepsilon
\end{aligned}
$$

For this model, we all know how to interpret the fixed marginal effects

But now… Spatial Lag Model (SAR) (transformed variables)

$$
\begin{aligned}
&Child\ Poverty\ Rate = -0.143 \\
&+ (0.339)(Proportion\ Female\text{-}Headed\ HH) \\
&+ (0.517)(Unemployment\ Rate) \\
&+ (-0.110)(Higher\ Education\ Rate) \\
&+ (0.453)\underline{W(Child\ Poverty\ Rate)} \\
&+ \varepsilon
\end{aligned}
$$

avg. of “neighbor’s” child poverty rate

For this model, interpreting marginal effects is *much* more difficult

I’ll return to this model at the end of my presentation

Why more difficult?

Spatial Spillovers &
Spatial Feedbacks

recall expected values for SAR model…

$$\hat{y} = \left[I + \rho W + \rho^2 W^2 + \ldots \right] X\beta$$

and what does this mean?

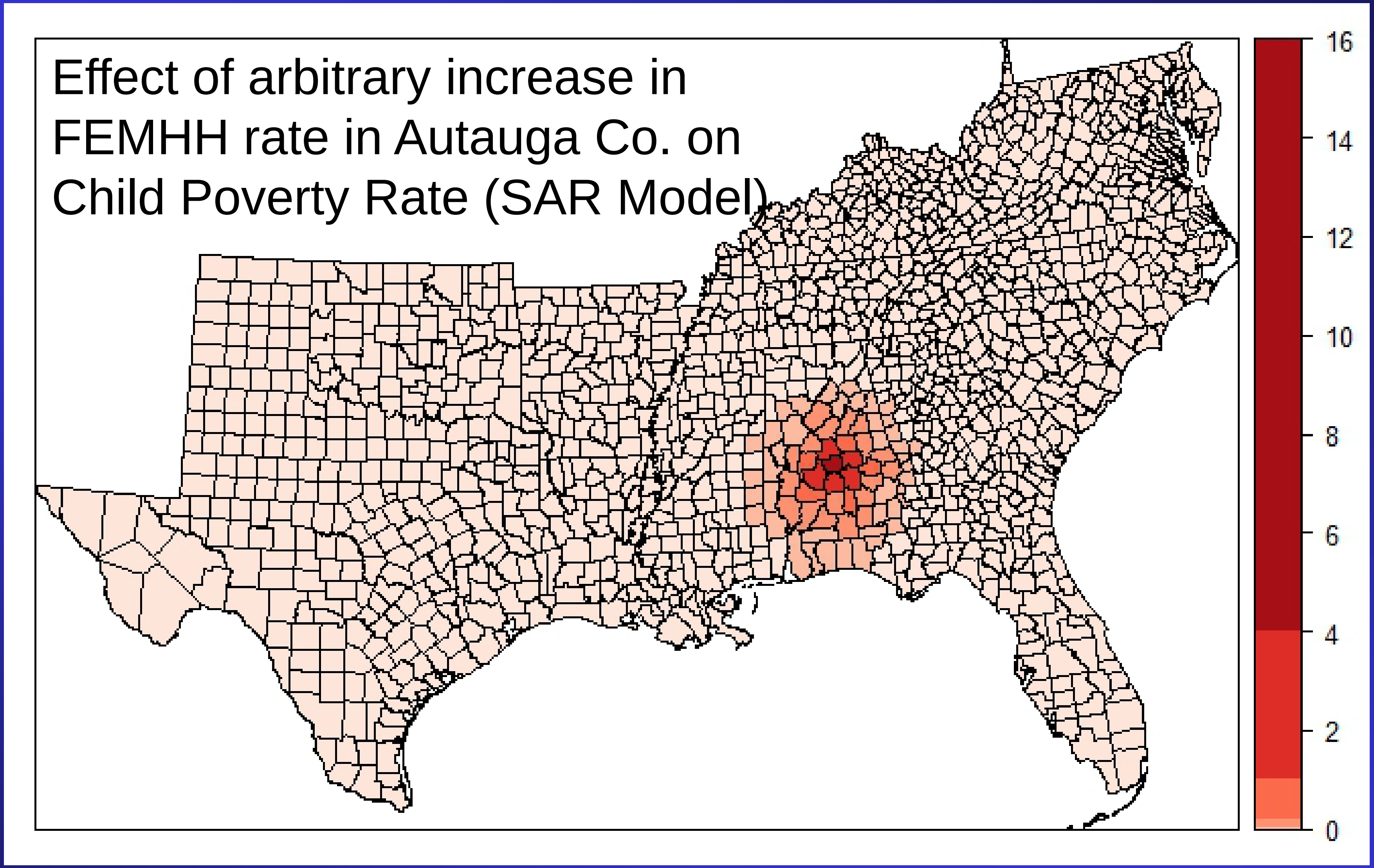


Illustration of “spatial spillover” on child poverty from a simulated increase in female-headed HH (with kids) only in Autauga Co. AL (SAR model with 1st-order queen *W*)

The next several slides attempt to show (mathematically) what’s happening here

Back to the Standard Linear Model (SLM)

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_i$$

In matrix notation: $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$

Q: For the standard linear model, what (*mathematically*) do the beta coefficients represent?

A: They represent the partial derivatives of y with respect to $\boldsymbol{x}_k$.

and a reminder: $\boldsymbol{\varepsilon} \sim \mathrm{N}_{\mathrm{iid}}(0,\sigma^2\boldsymbol{I})$

Think of it like this…

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_i$$

In matrix notation: $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$

$$E[y \mid X] = E[(X\beta + \varepsilon) \mid X] = X\beta$$

one of these for each β coefficient; i.e., one for each x_k

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \ldots & \frac{\partial y_1}{\partial x_{nk}} \\ \ldots & \ldots & \ldots \\ \frac{\partial y_n}{\partial x_{1k}} & \ldots & \frac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = \begin{pmatrix} \beta_k & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \beta_k \end{pmatrix} = \beta_k \begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix} = \beta_k I_n$$

$(n \times n)$ $(n \times n)$ $(n \times n)$

That is…

row 1: change in DV in region 1 based on change in x_{ik} each region (i = 1,…,n)

$$\begin{pmatrix} \boxed{\dfrac{\partial y_1}{\partial x_{1k}} \quad \cdots \quad \dfrac{\partial y_1}{\partial x_{nk}}} \\ \cdots \quad \cdots \quad \cdots \\ \dfrac{\partial y_n}{\partial x_{1k}} \quad \cdots \quad \dfrac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = \begin{pmatrix} \beta_k & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \beta_k \end{pmatrix} = \beta_k \begin{pmatrix} 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 1 \end{pmatrix} = \beta_k I_n$$

$$\frac{\partial y_j}{\partial x_{ik}} = \beta_k, \quad \text{for } i = j$$

$$\frac{\partial y_j}{\partial x_{ik}} = 0, \quad \text{for } i \neq j$$

cross-partial derivatives are zero; change in x_{ik} affects only y_i; no spatial spillovers

and…

$$\begin{pmatrix} \boxed{\begin{matrix} \dfrac{\partial y_1}{\partial x_{1k}} \\ \cdots \\ \dfrac{\partial y_n}{\partial x_{1k}} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \dfrac{\partial y_1}{\partial x_{nk}} \\ \cdots \\ \dfrac{\partial y_n}{\partial x_{nk}} \end{matrix} \end{pmatrix} = \begin{pmatrix} \beta_k & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_k \end{pmatrix} = \beta_k \begin{pmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix} = \beta_k I_n$$

column 1: change in DV in each region i $(i = 1,\ldots,n)$ based on change in x_{1k} (i.e., x_k in region 1)

$$\frac{\partial y_j}{\partial x_{ik}} = \beta_k, \quad \text{for } i = j$$

$$\frac{\partial y_j}{\partial x_{ik}} = 0, \quad \text{for } i \neq j$$

cross-partial derivatives are zero; change in x_{ik} affects only y_i; no spatial spillovers

Simple example…

Region 1 | Region 2 | Region 3

Assume:

- 3 regions; independent observations
- Standard linear model (SLM) estimated by OLS
- Dependent variable y ; Independent variable x_1
- Slope coefficient for x_1, β_1 = 2

Partial derivatives matrix for x_1:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{11}} & \frac{\partial y_1}{\partial x_{21}} & \frac{\partial y_1}{\partial x_{31}} \\ \frac{\partial y_2}{\partial x_{11}} & \frac{\partial y_2}{\partial x_{21}} & \frac{\partial y_2}{\partial x_{31}} \\ \frac{\partial y_3}{\partial x_{11}} & \frac{\partial y_3}{\partial x_{21}} & \frac{\partial y_3}{\partial x_{31}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Simple example (continued)...

Region 1

Region 2

Region 3

Now, further assume that variable x_1 in Region 3 changes by 10 units:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SLM} \approx \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \Delta x_{11} \\ \Delta x_{21} \\ \Delta x_{31} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}$$

Only the value of y_3 is affected by the change in x_3; no spatial spillovers

Again… things are more complex for SAR model (spatial spillovers!)

$$y = X\beta + \rho W y + \varepsilon$$

$$y = (I - \rho W)^{-1}(X\beta + \varepsilon)$$

$$E[y \mid X] = E[(I - \rho W)^{-1}(X\beta + \varepsilon)]$$

$$= (I - \rho W)^{-1} E[X\beta + \varepsilon]$$

$$= (I - \rho W)^{-1} X\beta$$

Now the partial derivative matrix looks like this…

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \cdots & \frac{\partial y_1}{\partial x_{nk}} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_{1k}} & \cdots & \frac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = (I - \rho W)^{-1} \begin{pmatrix} \beta_k & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_k \end{pmatrix} = \beta_k (I - \rho W)^{-1}$$

The n x n matrix $(I\text{-}\rho W)^{-1}$ has non-zero elements off the main diagonal. These non-zero cross-partial derivatives imply the existence of spatial spillovers. This follows from the power series approximation shown above:

$$(I - \rho W)^{-1} = I + \rho W + \rho^2 W^2 + \rho^3 W^3 + \ldots$$

With W row-standardized, the elements of W lie between 0 & 1. Further, under positive spatial autocorrelation, ρ is constrained to be strictly < |1|. Thus, the spatial spillovers associated with higher powers of ρ & W dampen out, often quickly.

Again, assume:

Further, assume the row-standardized W matrix:

$$W = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

For this particular weights matrix, we have (by simple matrix arithmetic):

$$(I-\rho W)=\begin{pmatrix} 1 & -\rho & 0 \\ -\rho/2 & 1 & -\rho/2 \\ 0 & -\rho & 1 \end{pmatrix}$$

and its inverse:

$$(I-\rho W)^{-1}=\left(\frac{1}{1-\rho^2}\right)\begin{pmatrix} 1-\rho^2/2 & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1-\rho^2/2 \end{pmatrix}$$

Thus:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \frac{\partial y_1}{\partial x_{2k}} & \frac{\partial y_1}{\partial x_{3k}} \\ \frac{\partial y_2}{\partial x_{1k}} & \frac{\partial y_2}{\partial x_{2k}} & \frac{\partial y_2}{\partial x_{3k}} \\ \frac{\partial y_3}{\partial x_{1k}} & \frac{\partial y_3}{\partial x_{2k}} & \frac{\partial y_3}{\partial x_{3k}} \end{pmatrix} = \left(\frac{\beta_k}{1-\rho^2} \right) \begin{pmatrix} \left(1-\rho^2/2\right) & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1-\rho^2/2 \end{pmatrix}$$

The n x n matrix of partial derivatives is a function of the exogenously specified weights matrix, W, the spatial scalar parameter, ρ , and the parameter β_k. But the β-term appears not only along the major diagonal but also in the cross-partial derivatives. When $\rho = 0$, the matrix of partial derivatives is the one we saw for the SLM. When $\rho \neq 0$, the partial derivatives (diagonal) are greater than the β-term, augmented by spatial spillovers as follows:

Diagonal terms of the partial derivatives matrix (for our 3-region example and the specified 1st-order weights matrix, *W*):

$$\frac{\partial y_1}{\partial x_{1k}} = \frac{\partial y_3}{\partial x_{3k}} = \left(\frac{\beta_k}{1-\rho^2}\right)\left(1-\rho^2/2\right) = \beta_k\left(\frac{2-\rho^2}{2-2\rho^2}\right) > \beta_k$$

$$\frac{\partial y_2}{\partial x_{2k}} = \left(\frac{\beta_k}{1-\rho^2}\right) > \beta_k$$

Again, when $\rho = 0$, the diagonal partial derivatives are simply the β-terms. When $\rho \neq 0$, the diagonal elements represent the β-terms, augmented by spatial spillovers

Simple example revisited…

Region 1 | Region 2 | Region 3

Assume:

- 3 regions; independent observations
- SAR model
- Dependent variable y ; Independent variable x_1
- Slope coefficient for x_1, β_1 = 2

Partial derivatives matrix for x_1:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{11}} & \frac{\partial y_1}{\partial x_{21}} & \frac{\partial y_1}{\partial x_{31}} \\ \frac{\partial y_2}{\partial x_{11}} & \frac{\partial y_2}{\partial x_{21}} & \frac{\partial y_2}{\partial x_{31}} \\ \frac{\partial y_3}{\partial x_{11}} & \frac{\partial y_3}{\partial x_{21}} & \frac{\partial y_3}{\partial x_{31}} \end{pmatrix} = \left(\frac{2}{1-\rho^2}\right) \begin{pmatrix} \left(1-\rho^2/2\right) & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1-\rho^2/2 \end{pmatrix}$$

Simple example (continued) …

Region 1	Region 2	Region 3

Further, assume an increase by 10 units the value of x_1 in Region 3:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix} \approx \left(\frac{2}{1-\rho^2} \right) \begin{pmatrix} \left(1-\rho^2/2\right) & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1-\rho^2/2 \end{pmatrix} \begin{pmatrix} \Delta x_{11} = 0 \\ \Delta x_{21} = 0 \\ \Delta x_{31} = 10 \end{pmatrix} = \left(\frac{20}{1-\rho^2} \right) \begin{pmatrix} \rho^2/2 \\ \rho/2 \\ 1-\rho^2/2 \end{pmatrix}$$

$$\begin{pmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{pmatrix} \approx \left(\frac{2}{1-\rho^2}\right)\begin{pmatrix}\left(1-\rho^2/2\right) & \rho & \rho^2/2\\ \rho/2 & 1 & \rho/2\\ \rho^2/2 & \rho & 1-\rho^2/2\end{pmatrix}\begin{pmatrix}\Delta x_{11}=0\\ \Delta x_{21}=0\\ \Delta x_{31}=10\end{pmatrix} = \left(\frac{20}{1-\rho^2}\right)\begin{pmatrix}\rho^2/2\\ \rho/2\\ 1-\rho^2/2\end{pmatrix}$$

When $\rho = 0$, we obtain the earlier results for the SLM (i.e., increase of 10 units for variable x_1 in Region 3 results in y_3 increasing by 20 units):

$$\begin{pmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{pmatrix}_{\substack{SLM\\ or SAR\\ (\rho=0)}} \approx \left(\frac{20}{1}\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 20\end{pmatrix}$$

$$\begin{pmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{pmatrix} \approx \left(\frac{2}{1-\rho^2}\right)\begin{pmatrix}\left(1-\rho^2/2\right) & \rho & \rho^2/2\\ \rho/2 & 1 & \rho/2\\ \rho^2/2 & \rho & 1-\rho^2/2\end{pmatrix}\begin{pmatrix}\Delta x_{11}=0\\ \Delta x_{21}=0\\ \Delta x_{31}=10\end{pmatrix} = \left(\frac{20}{1-\rho^2}\right)\begin{pmatrix}\rho^2/2\\ \rho/2\\ 1-\rho^2/2\end{pmatrix}.$$

If, however, $\rho = 0.2$, then we see that an increase of 10 units for variable x_1 in Region 3 results in y increasing in all regions:

$$\begin{pmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{pmatrix}_{SAR(0.2)} \approx \left(\frac{20}{1-(0.2)^2}\right)\begin{pmatrix}(0.2)^2/2\\ (0.2)/2\\ 1-(0.2)^2/2\end{pmatrix} = \begin{pmatrix}0.42\\ 2.08\\ 20.42\end{pmatrix}.$$

Can we describe this outcome?

$$\begin{pmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{pmatrix}_{SAR(0.2)} \approx \left(\frac{20}{1-(0.2)^2}\right)\begin{pmatrix}(0.2)^2/2\\ (0.2)/2\\ 1-(0.2)^2/2\end{pmatrix} = \begin{pmatrix}0.42\\ 2.08\\ 20.42\end{pmatrix}$$

y increases everywhere because of spatial spillovers. The spillover effect is strongest for Region 2 (an immediate neighbor of Region 3 (which we might have anticipated because of the 1st-order spatial weights matrix, W). But Region 1 is a "neighbor of the neighbor" and also is affected by the change in Region 3 for this SAR model.

But why did y in Region 3 increase by more than 20? The answer is feedback spillover. A portion of the change in the other regions is feeding back to further change y_3.

Calculate the extent of spatial spillover for this simple SAR model by subtracting the results for $\rho = 0.0$ from the results for $\rho = 0.2$

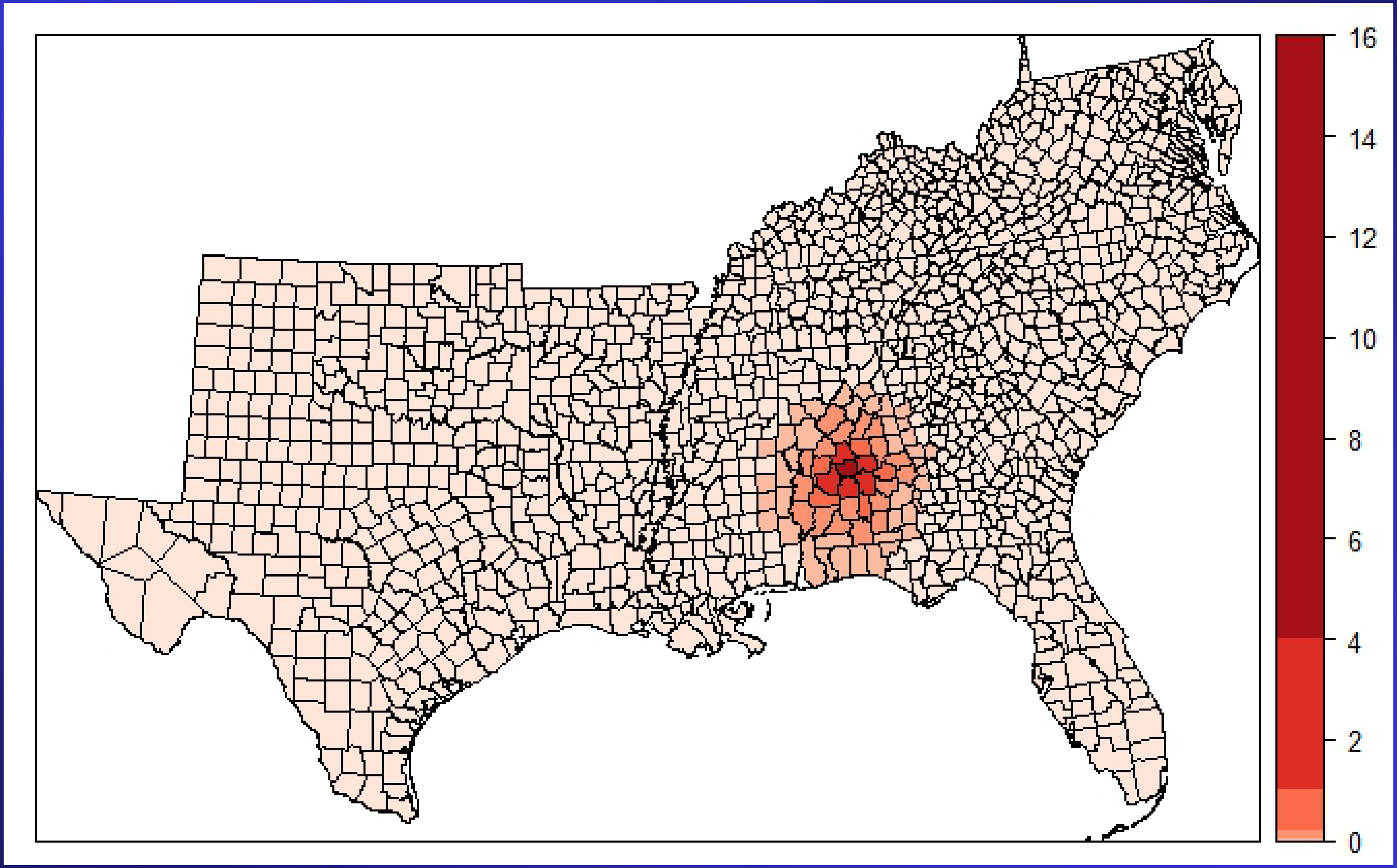
$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SAR(\rho=0.2)} - \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{(\rho=0.0)} = \begin{pmatrix} 0.42 \\ 2.08 \\ 0.42 \end{pmatrix}$$

The change in Region 3 spills over to affect Region 2 and, much more mildly, even Region 1. Since Region 3 is also a neighbor or Region 2, the same degree of spillover comes back to Region 3 as feedback spillover

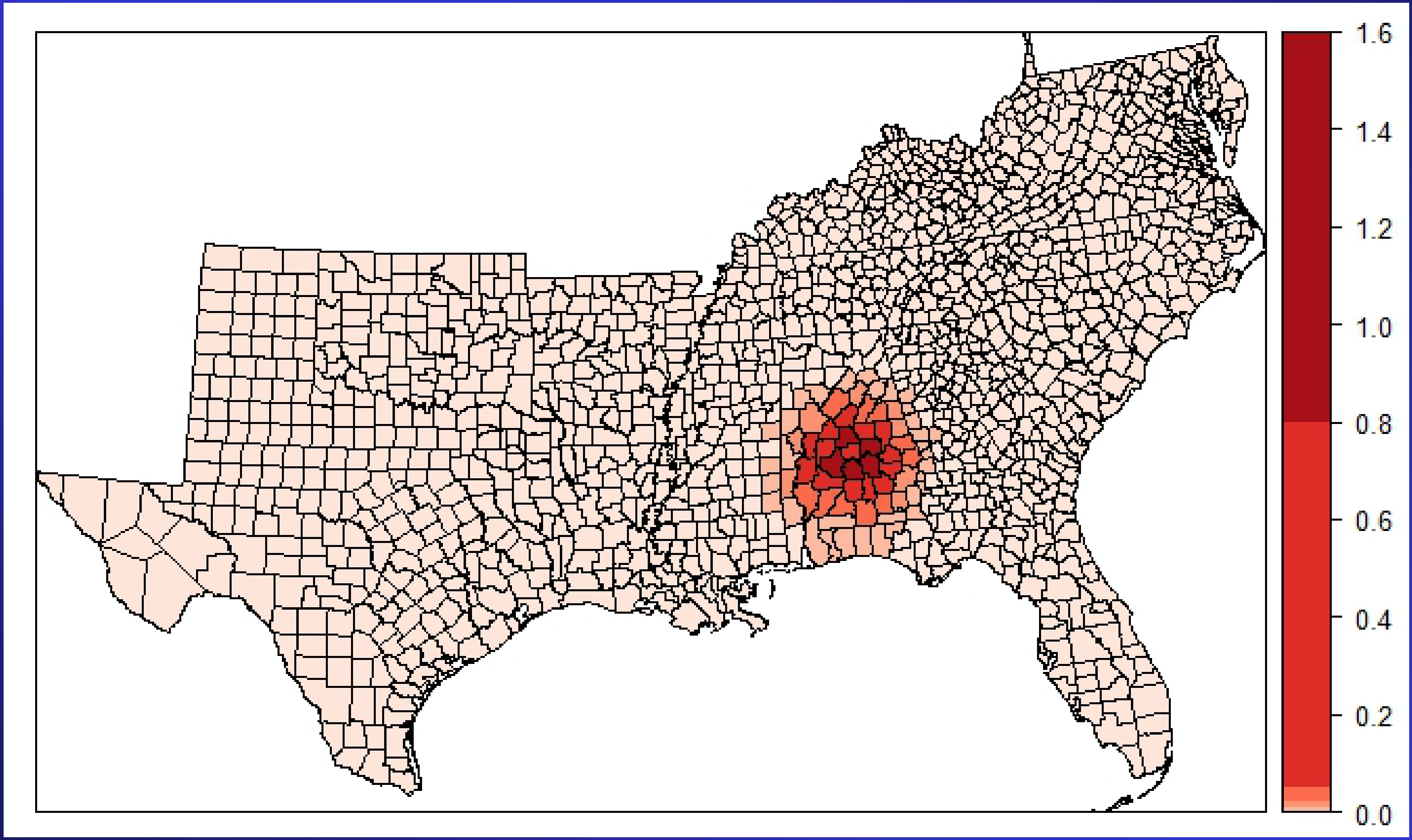
For the southeastern counties, and for the simple model & estimated parameters shown earlier (3 independent variables plus spatial lag)…

Employing the SAR model, we chose arbitrarily to increase the independent variable FEMHH in Autauga County AL by 20 percentage points (from 19% to 39%) and inquired about the impact of this change on county-level poverty rates

Change in child poverty rate due to simulated increase in FEMHH variable in Autauga Co.

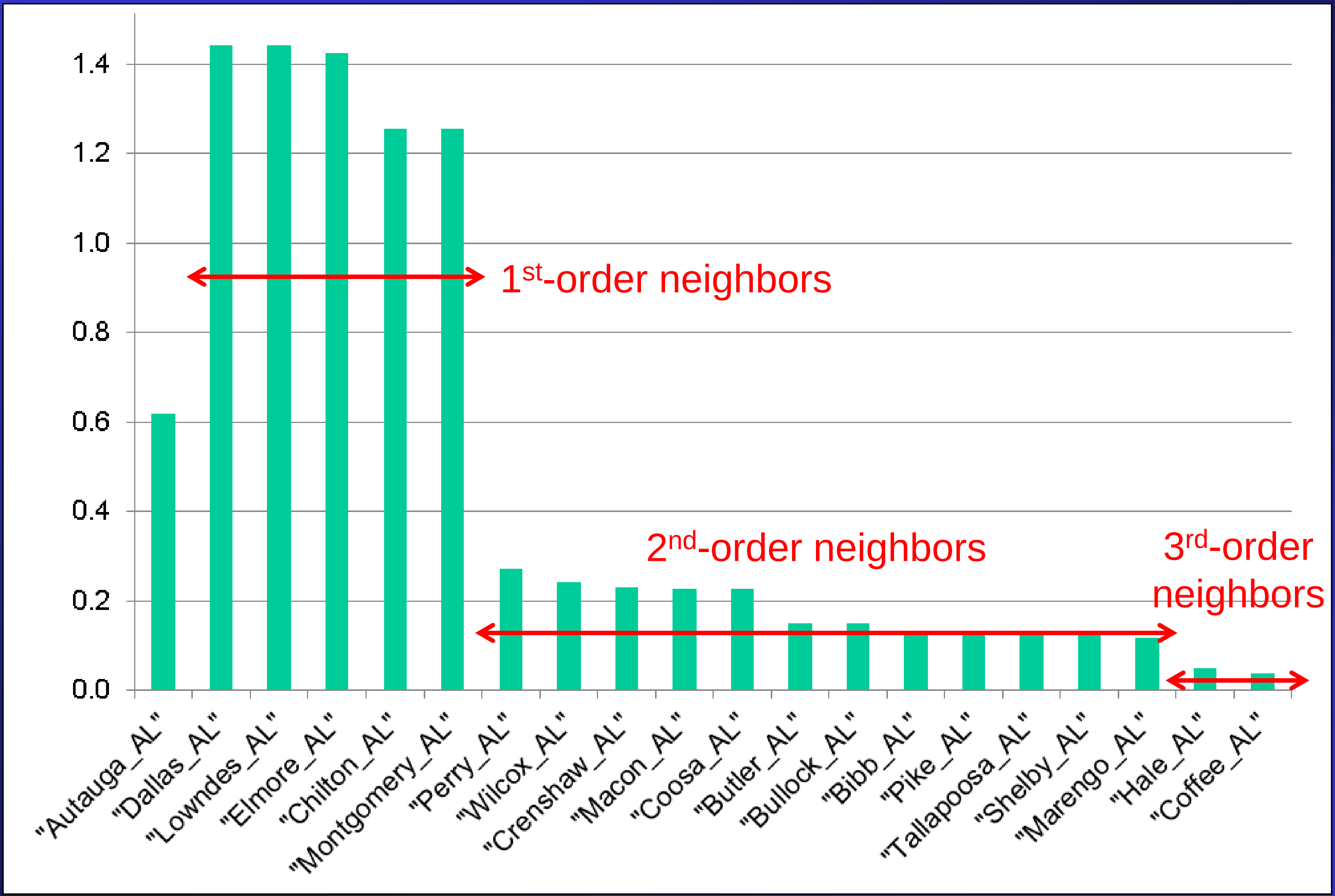


Spatial spillovers due to change in child poverty rate due to simulated increase in FEMHH variable in Autauga Co. (direct effect removed)



Region	Modifications in explanatory variables			Poverty levels – Original values (%)	Poverty levels - Updated values (%)	Total changes in poverty levels (%)	Changes in poverty levels due to spillovers (%)
	FEM	UNEM	HS				
Autauga Co.	20%	0%	0%	13.7	29.5	15.80	0.62
Dallas Co.	0%	0%	0%	41.0	42.4	1.44	1.44
Lowndes Co	0%	0%	0%	41.8	43.2	1.44	1.44
Elmore Co.	0%	0%	0%	14.4	15.8	1.42	1.42
Chilton Co.	0%	0%	0%	19.9	21.1	1.26	1.26
Montgomery Co	0%	0%	0%	25.3	26.5	1.25	1.25
Perry Co	0%	0%	0%	49.2	49.5	0.27	0.27
Wilcox Co.	0%	0%	0%	48.5	48.7	0.24	0.24
Crenshaw Co.	0%	0%	0%	28.7	29.0	0.23	0.23
Macon Co.	0%	0%	0%	44.1	44.4	0.23	0.23
Coosa Co.	0%	0%	0%	19.5	19.8	0.23	0.23
Butler Co.	0%	0%	0%	31.6	31.8	0.15	0.15
Bullock Co.	0%	0%	0%	45.0	45.1	0.15	0.15
Bibb Co.	0%	0%	0%	28.1	28.2	0.13	0.13
Pike Co.	0%	0%	0%	30.0	30.1	0.13	0.13
Tallapoosa Co.	0%	0%	0%	24.6	24.7	0.13	0.13
Shelby Co.	0%	0%	0%	7.4	7.5	0.13	0.13
Marengo Co.	0%	0%	0%	33.9	34.0	0.12	0.12
Hale Co.	0%	0%	0%	34.1	34.2	0.05	0.05
Coffee Co.	0%	0%	0%	22.5	22.6	0.04	0.04
…	…	…	…	…	…	…	…

Graph of spatial spillovers



Comments…

- Interpreting marginal effects in spatial regression models becomes complicated
- They are influenced by effects direct & indirect (spatial spillovers & feedbacks)
- They are a function of:
  - Our data and estimated parameters (β_k & ρ)
  - Our assumed neighborhood definition and spatial weights matrix, W
- An example such as shown here is helpful for understanding what's going on in these models but requires some cautions